

Para-Liouville/Toda central charges from M5-branes

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abstract

We propose that N M5-branes, put on $\mathbb{R}^4/\mathbb{Z}_m$ with deformation parameters $\epsilon_{1,2}$, realize two-dimensional theory with $\widehat{\text{SU}}(m)_N$ symmetry and m -th para- W_N symmetry. This includes the standard W_N symmetry for $m = 1$ and super-Virasoro symmetry for $m = N = 2$. We provide a small check of this proposal by calculating the central charge of the 2d theory from the anomaly polynomial of the 6d theory.

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Introduction: N M5-branes, put on \mathbb{R}^4 with Nekrasov's deformation parameters $\epsilon_{1,2}$, are now believed to realize two-dimensional theory with W_N symmetry [1, 2]. One check of this statement was given in [3] following the observation made in [4]. Namely, the equivariant integral on \mathbb{R}^4 of the anomaly polynomial of N M5-branes determined in [5] gives the anomaly polynomial of the 2d theory, from which the central charge of the Toda theory with W_N symmetry can be reproduced. The same analysis can be performed for 6d $\mathcal{N} = (2, 0)$ theory of type $G = A, D, E$ whose anomaly polynomial is also known [6, 7]; and it correctly reproduces the central charge of the Toda theory of type G .

In a recent paper [8], it was proposed that two M5-branes on $\mathbb{R}^4/\mathbb{Z}_2$ with deformations $\epsilon_{1,2}$ give rise to a system with the $\widehat{\text{SU}}(2)_2$ symmetry and the super-Virasoro symmetry. As a generalization, we propose that N M5-branes on $\mathbb{R}^4/\mathbb{Z}_m$ realize a 2d system with a free boson, $\widehat{\text{SU}}(m)_N$, and the m -th para- W_N symmetry. Here \mathbb{Z}_m acts as $(z, w) \mapsto (e^{2\pi/m}z, e^{-2\pi/m}w)$ on $(z, w) \in \mathbb{C}^2 \simeq \mathbb{R}^4$. We give a small piece of supporting evidence by calculating the central charge from the 6d anomaly polynomial. We will also speculate what happens if the 6d $\mathcal{N} = (2, 0)$ theory of type G is used instead. In the following G stands for one of A_n, D_n or E_n ; r_G, h_G and d_G are the rank, the (dual) Coxeter number and the dimension of G , respectively. They satisfy $d_G = r_G(h_G + 1)$.

Para- W symmetry and para-Toda theory: One way to realize the $W(\widehat{G})$ symmetry [9, 10, 11, 12, 13, 14] is to consider the chiral algebra of the coset

$$\widehat{G}_k \times \widehat{G}_m / \widehat{G}_{k+m} , \quad (1)$$

for $m = 1$. The m -th para- $W(\widehat{G})$ symmetry is obtained by taking an arbitrary positive integer m in this coset; in particular, it reduces to the super-Virasoro algebra when $m = N = 2$. Generalization of NSR superstrings using these algebras were explored e.g. in [15, 16].

The m -th para- $W(\widehat{G})$ algebra is the symmetry of the m -th para-Toda model of type G , which has the following action [17]

$$S = S \left(\frac{\widehat{G}_m}{\widehat{\text{U}}(1)^{r_G}} \right) + \int d^2x \left[\partial_\mu \Phi \partial_\mu \Phi + \sum_{i=1}^{r_G} \Psi_i \bar{\Psi}_i \exp \left(\frac{b}{\sqrt{m}} \alpha_i \cdot \Phi \right) \right] . \quad (2)$$

Here, $\widehat{G}_m/\widehat{\text{U}}(1)^{r_G}$ describes the generalized parafermions Ψ_i of type G [18], α_i are simple roots of G , Φ are r_G free bosons with background charge $(b + 1/b)\rho/\sqrt{m}$ with the Weyl vector ρ . The central charge is given by

$$c = c \left(\frac{\widehat{G}_m}{\widehat{\text{U}}(1)^{r_G}} \right) + r_G + \frac{h_G d_G}{m} \left(b + \frac{1}{b} \right)^2 = \frac{m d_G}{m + h_G} + \frac{h_G d_G}{m} \left(b + \frac{1}{b} \right)^2 . \quad (3)$$

Note that the parafermion Ψ_i has dimension $1 - 1/m$, and the exponential of bosons has dimension $1/m$, so that the interaction terms are marginal. For $m = 1$ this is the usual affine Toda theory, and for $m = N = 2$ this is the $\mathcal{N} = 1$ super-Liouville theory.

N M5-branes on $\mathbb{R}^4/\mathbb{Z}_m$: The anomaly polynomial I_8 of 6d $\mathcal{N} = (2, 0)$ is given by the general form

$$I_8 = \mathcal{A} I_8(1) + \mathcal{B} p_2(NW)/24 , \quad (4)$$

where $I_8(1)$ is the anomaly polynomial of a single M5-brane, NW is the normal bundle to the worldvolume W of the 6d theory, and \mathcal{A}, \mathcal{B} are integers determined by the type of the 6d theory. For N M5-branes, $\mathcal{A} = N$ and $\mathcal{B} = N^3 - N$. When compactified on a four-manifold X_4 with a suitable twist, it was determined in [3] that the resulting 2d theory has the central charge

$$c = \chi(X_4)\mathcal{A} + (P_1(X_4) + 2\chi(X_4))\mathcal{B} . \quad (5)$$

Here $\chi(X_4)$ and $P_1(X_4)$ are the Euler number and three times the signature of X_4 , respectively. We let X_4 be $\mathbb{R}^4/\mathbb{Z}_m$ with the deformations $\epsilon_{1,2}$. Then χ and P_1 are to be taken in the equivariant sense, and are given by¹

$$\chi(X_4) = m , \quad P_1(X_4) = \frac{1}{m} \frac{(\epsilon_1 + \epsilon_2)^2}{\epsilon_1 \epsilon_2} - 2m . \quad (6)$$

Therefore, N M5-branes on $\mathbb{R}^4/\mathbb{Z}_m$ give rise to a 2d system with the central charge

$$c = Nm + \frac{N^3 - N}{m} \left(b + \frac{1}{b} \right)^2 , \quad (7)$$

where we used the standard identification $\epsilon_1/\epsilon_2 = b^2$.

According to the general lore, the Hilbert space of the 2d theory comes from the BPS states of the supersymmetric quantum mechanics on the moduli space of instantons on X_4 . In our case, we consider $U(N)$ instantons on $\mathbb{R}^4/\mathbb{Z}_m$, for which it is known that there is an action of a free boson and of $\widehat{SU}(m)_N$; this was found in [19, 20] and string

¹The calculation is done as follows. Let us parameterize $\mathbb{R}^4 \simeq \mathbb{C}^2$ by (z, w) , on which two rotations act via $(z, w) \mapsto (e^{\epsilon_1} z, e^{\epsilon_2} w)$. On the blowup of $\mathbb{R}^4/\mathbb{Z}_m$, we have m fixed points of $U(1)^2$ actions, whose local coordinates are given by $(z_i, w_i) = (z^{m-i+1} w^{1-m}, z^{i-m} w^m)$ for $i = 1, \dots, m$. Let us define $\epsilon_{1,2}(i)$ by the $U(1)^2$ action at the fixed points: $(z_i, w_i) \mapsto (e^{\epsilon_1(i)} z_i, e^{\epsilon_2(i)} w_i)$. Then the topological numbers are given by the fixed point formula: $\chi = \sum_i 1$, and $P_1 = \sum_i (\epsilon_1(i)^2 + \epsilon_2(i)^2)/(\epsilon_1(i)\epsilon_2(i))$.

theory interpretation was later given e.g. in [21]. Then, the central charge (7) needs to be subdivided to

$$c = 1 + c\left(\widehat{\text{SU}}(m)_N\right) + \left[\frac{m(N^2 - 1)}{m + N} + \frac{N^3 - N}{m} \left(b + \frac{1}{b}\right)^2\right]. \quad (8)$$

The third term is the central charge of the m -th para-Toda theory of type $\text{SU}(N)$ we found in Eq. (3).

We interpret this calculation as a check to the proposal that N M5-branes on $\mathbb{R}^4/\mathbb{Z}_m$ with deformations $\epsilon_{1,2}$ give rise to a 2d system with actions of a free boson, $\widehat{\text{SU}}(m)_N$ and m -th para- W_N algebra with central charge (3). This statement reduces to the now-standard relations in [1, 2] when $m = 1$, and to the proposal in [8] when $m = N = 2$.

Speculation concerning 6d theory of general type on $\mathbb{R}^4/\mathbb{Z}_m$: First let us consider what happens if we start from 6d $\mathcal{N} = (2, 0)$ theory of type A_{N-1} , instead of N M5-branes. One needs to decouple the center-of-mass mode, which changes \mathcal{A}, \mathcal{B} in Eqs. (4), (5) to $\mathcal{A} = N - 1$ and $\mathcal{B} = N^3 - N$. The resulting 2d theory has the central charge of the form

$$c = c\left(\frac{\widehat{\text{SU}}(m)_N}{\widehat{\text{U}}(1)^{m-1}}\right) + \left[c\left(\frac{\widehat{\text{SU}}(N)_m}{\widehat{\text{U}}(1)^{N-1}}\right) + (N - 1) + \frac{N^3 - N}{m} \left(b + \frac{1}{b}\right)^2\right]. \quad (9)$$

We see two cosets realizing generalized parafermions, known also as $\text{RCFT}[A_{m-1}, A_{N-1}]$ and $\text{RCFT}[A_{N-1}, A_{m-1}]$ in the terminology of [22], respectively. In general, $\text{RCFT}[\Gamma, G]$ for $\Gamma, G = A, D, E$ is a rational CFT with central charge

$$c(\text{RCFT}[\Gamma, G]) = \frac{h_\Gamma r_\Gamma r_G}{h_\Gamma + h_G}. \quad (10)$$

Note that $\text{RCFT}[G, A_{m-1}]$ is the generalized parafermion of type G , but that $\text{RCFT}[\Gamma, G]$ is not yet constructed for general pair of G, Γ . We have

$$c(\text{RCFT}[\Gamma, G]) + c(\text{RCFT}[G, \Gamma]) = r_\Gamma r_G, \quad (11)$$

and it is believed they comprise a ‘level-rank-dual’ pair of RCFTs.

The 6d theory of type G has $\mathcal{A} = r_G$ and $\mathcal{B} = d_G h_G$ in Eqs. (4), (5) [6, 7]. Using $d_G = r_G(h_G + 1)$, we find that the 6d theory on $\mathbb{R}^4/\mathbb{Z}_m$ has the central charge

$$c = c(\text{RCFT}[A_{m-1}, G]) + \left[c(\text{RCFT}[G, A_{m-1}]) + r_G + \frac{d_G h_G}{m} \left(b + \frac{1}{b}\right)^2\right], \quad (12)$$

where the second term is the central charge of the m -th para-Toda theory of type G we saw in Eq. (3). This suggests that the resulting 2d theory has the symmetry $\text{RCFT}[A_{m-1}, G]$ and the m -th para- $W(\widehat{G})$ symmetry. Note that $\text{RCFT}[A_{m-1}, G]$ is not yet constructed when $G \neq A$.

We can further generalize the system by considering instantons of gauge group $G = A, D, E$ on the ALE orbifold of type Γ . Nekrasov's deformation cannot be performed when $\Gamma \neq A$, because the ALE orbifold of type D and E does not have $U(1)^2$ isometry. We can still expect that this construction might naturally give us $\text{RCFT}[\Gamma, G] + \text{RCFT}[G, \Gamma]$. The symmetry under G and Γ can be understood once one realizes that the 6d theory of type G is Type IIB string on the ALE orbifold of type G . Then, the 2d system is Type IIB string on the ALE orbifold of type G times the ALE orbifold of type Γ , which is manifestly symmetric under the exchange of G and Γ .

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